Inhomogeneous vortex matter

J.R. Anglin[1] and M. Crescimanno[2]
[1]*M.I.T.-Harvard Center for Ultracold Atoms, 77 Massachusetts Ave., Cambridge MA 02139,*
[2]*Center for Photon Induced Processes, Physics Department,*
Youngstown State University, Youngstown, OH 12345.
(Dated: February 1, 2008)

We present a generalization of the continuum theory of vortex matter for non-uniform superfluid density. This theory explains the striking regularity of vortex lattices observed in Bose-Einstein condensates, and predicts the frequencies of long-wavelength lattice excitations.

Dense lattices of quantized vortices in rotating Bose-Einstein condensates (BECs) [1, 2] are strikingly more regular than finite vortex arrays in homogeneous superfluid [3] (see Fig. 1), even though BEC densities vary greatly over the sample. This Letter generalizes the Feynman-Tkachenko [4, 5] continuum theory of 'vortex matter' to cases in which the condensate density varies slowly on the scale of the lattice spacing. This theory explains the lattices' surprising regularity, and find pronounced effects of nonuniform density on lattice excitations.

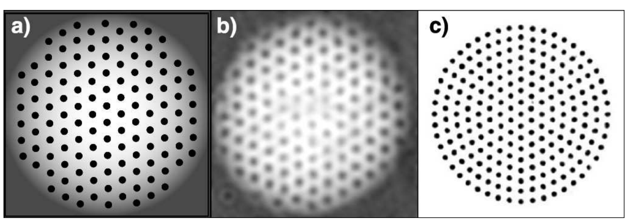

FIG. 1: (a) Static lattice according to Eq.(14), translated and rotated to match (b) experimental data courtesy of J.R. Abo-Shaeer. Compare (c) vortex array in constant ρ (217_2 from Fig. 5 of Ref. [3]).

We consider a two-dimensional regular array of vortices, whether realized in a very oblate BEC, or as parallel vortex lines in a prolate one. Denoting the lattice length scale by b and introducing dimensionless complex coordinates $bz = x + iy$, a regular lattice has the positions z_{jk} of parallel vortex lines given by $z_{jk} = z^0_{jk} \equiv k + \tau j$ for $\tau = \tau_1 + i\tau_2$ ($\tau_2 > 0$ and τ_i real). Much is known about vortex lattices in superfluids of constant density ρ, simply from incompressible hydrodynamics; and it is all simplified by using dimensionless variables, expressing time, velocity, and energy in lattice units $Mb^2/\hbar$, $\hbar/(Mb)$, $\hbar^2/(Mb^2)$ respectively, for M the mass of the particles composing the superfluid. The regular triangular case $\tau = (1 + i\sqrt{3})/2$ is the ground state [5] of a sample rotating at dimensionless rate $\Omega = \pi/\tau_2$. The irrotational velocity field $v \equiv v_x + iv_y$ consists of a fine field v_f, which is periodic on the lattice scale, plus a coarse field v_c obeying Feynman's criterion [4]

$$\partial_z v_c - \partial_{\bar z} \bar v_c = 2\pi i \rho_V \tag{1}$$

where for a regular lattice the vortex density ρ_V is $1/\tau_2$. (In our complex notation $2\partial_z A = b(\vec\nabla \cdot \vec A + i\vec\nabla \times \vec A)$ for any A. We assume counter-clockwise rotation.) And long-wavelength excitations of the lattice, $z_{jk}(t) \to z^0_{jk} + D\left(z^0_{jk}, \bar z^0_{jk}; t\right)$ obey the wave equations [5]

$$i\partial_t \left(\partial_z D - \partial_{\bar z} \bar D\right) = 2\Omega \left(\partial_z D + \partial_{\bar z} \bar D\right) \tag{2}$$

$$i\partial_t \left(\partial_z D + \partial_{\bar z} \bar D\right) = -\frac{1}{2}\partial_{\bar z z} \left(\partial_z D - \partial_{\bar z} \bar D\right). \tag{3}$$

(Extensions to three-dimensional vortex matter [6, 7] become considerably more complicated.)

Although current dilute gaseous BECs are compressible fluids governed by the Gross-Pitaevskii equation (GPE) [8], much of BEC vortex physics can be cast into a simpler hydrodynamic form. In our dimensionless variables, the GPE in a frame rotating about the co-ordinate origin may be written

$$\partial_t \rho = \Omega \partial_\phi \rho - \left[\partial_z (\rho v) + \partial_{\bar z} (\rho \bar v)\right] \tag{4}$$

$$\partial_t v = \partial_{\bar z} \left[|v - i\Omega z|^2 + 2V + 2g\rho - \frac{\partial_{\bar z z}\sqrt{\rho}}{\sqrt{\rho}}\right] \tag{5}$$

$$V \equiv V_{trap} - \frac{\Omega^2}{2}|z|^2$$

where v is still the lab-frame velocity, ϕ is the polar angular co-ordinate, $V_{trap}(z, \bar z)$ is the trap potential, and g is the dimensionless 2D coupling, determined by atomic and trap parameters. Except very near vortex cores, the rotating-frame velocity $v - i\Omega z$ is of order unity; and $\rho^{-1/2}\partial_{\bar z z}\sqrt{\rho}$ is no larger, except in cores and sample edges, which may be treated separately as boundary layers. In current experiments the *healing length* $\xi \equiv (g\rho)^{-1/2}$, which sets the vortex core size, is everywhere else much smaller than b.

So, outside vortex cores, the leading order results in the co-rotating frame are

$$\rho = \text{const.} - \frac{V}{g} \tag{6}$$

$$\Omega \partial_\phi \rho = \partial_z (\rho v) + \partial_{\bar z} (\rho \bar v). \tag{7}$$

Thus ρ depends on v only through the centrifugal modification of the trap potential. The velocity field, v is determined by (7), plus the condition of irrotationality except at the quantized vortex cores:

$$\partial_z v - \partial_{\bar{z}}\bar{v} = 2\pi i \sum_{jk} \delta^2 \left(z - z_{jk}\right). \tag{8}$$

This, with $\vec{\nabla}\cdot\vec{v} = 0$ instead of (7), is the starting point for Ref. [5]. So for constant V, and hence uniform ρ, the results of the incompressible case also apply to BECs.

If ρ varies slowly in space, will not inhomogeneous effects be small? Not obviously: like ρ itself, the lattice shape might vary slowly, but change greatly over the whole sample. Indeed, even for constant ρ in the ground state of a finite vortex array, Campbell and Ziff [3] found gradual but cumulatively large distortions. But there is a basic problem in extending their analysis to non-uniform ρ.

Investigations of homogeneous vortex matter have generally relied on the exact single-vortex solution $v = v_1(\bar{z} - \bar{z}_{jk})$ to (7) and (8) for constant ρ, which one may simply sum over the vortex labels j, k, because (7) and (8) are linear in v. For non-constant ρ the familiar $v_1(z) = i/\bar{z}$ satisfies (8) but not (7), and so we do not have the exact single-vortex solution. Perturbative approximations about $v_1(z) = i/\bar{z}$ as an ansatz break down at distances from the core beyond the length scale of the density variation [9]. So the few-vortex problem in inhomogeneous BECs becomes analytically intractable. For sufficiently dense lattices, however, inhomogeneous vortex matter yields to a different approach.

By a 'dense' vortex lattice, we mean $\rho = \rho(\varepsilon z, \varepsilon\bar{z})$ for small ε. (For a round harmonic trap with Thomas-Fermi radius R, $\varepsilon = b/R$, giving $\varepsilon \sim 0.1$ in current experiments.) We can therefore perturb in ε; but to distinguish smallness from slowness, we must use multiple scale analysis (MSA) [10]. This formalism produces coarse-scale equations of motion for D, from which all lattice-scale physics has been eliminated, in the same sense that high-frequencies are eliminated by adiabatic methods. These will be our generalizations of (7) and (8).

The application of MSA leads to a rather involved derivation the details of which will be reported elsewhere; here we outline its steps and report its conclusions. We begin by satisfying (7) identically by defining

$$v = i\Omega z + \frac{i}{\rho}\partial_{\bar{z}}\left(\rho F\right) \tag{9}$$

for real F. We use vortex-centered co-ordinates:

$$z = z' + D(z', \bar{z}', t), \tag{10}$$

so that $z'_{jk} = z^0_{jk}$ regardless of D. We do this so that in the z' co-ordinates we always have a regular lattice, whose symmetries we can exploit, even though the physical lattice may be distorted by excitations, or by a static D field induced by inhomogeneous ρ. In the non-Cartesian z' co-ordinates, Eqn. (8) becomes

$$\mathrm{Re}\left(\partial_{z'}\frac{\partial_{\bar{z}'} - 2\left(\partial_{\bar{z}'}D\right)\partial_{z'}}{\rho}\rho F\right) = \pi\sum_{jk}\delta^2\left(z' - z^0_{jk}\right) - \Omega\left(1 + \partial_{z'}D + \partial_{\bar{z}'}\bar{D} + \begin{vmatrix} \partial_{z'}D & \partial_{z'}\bar{D} \\ \partial_{\bar{z}'}D & \partial_{\bar{z}'}\bar{D} \end{vmatrix}\right)$$

This shows explicitly that only gradients in D affect F.

MSA then embeds the physical z'-plane in a fictitious 4-space of complex co-ordinates ζ, Z, as the subspace $(\zeta, Z) = (z', \varepsilon z')$. This provides $\partial_{z'} \to \partial_\zeta + \varepsilon\partial_Z$, etc., and proceeding perturbatively in ε, we are able to write explicit solutions (in terms of rapidly converging series) for the ζ-dependence of F at every order (we need to go to third). As usual with MSA, the 'gauge freedom' in how functions depend explicitly on the two extra dimensions is used to remove solutions growing secularly with ζ, by constraining the purely Z-dependent part of F (which we denote by F_c). Once we restrict back to physical two-space by setting $Z = \varepsilon z'$, $\zeta = z'$, and return from vortex-fixed z' to Cartesian z, we recognize the constraint on F_c as just what is needed to maintain Feynman's condition (1), for ρ_V as perturbed by D:

$$\partial_z\frac{\partial_{\bar{z}}\left(\rho F_c\right)}{\rho} + \partial_{\bar{z}}\frac{\partial_z\left(\rho F_c\right)}{\rho} = -2\Omega\left(\partial_z D + \partial_{\bar{z}}\bar{D}\right), \tag{11}$$

where we drop determinant terms quadratic in D (because we will have D order ε or smaller).

Having solved for F, and hence v, in terms of explicit lattice-periodic functions and v_c, we know the local fluid velocity near each vortex. This fixes the instantaneous vortex translational velocity field $\dot{D}$; but the fixing is not trivial. Since the hydrodynamic approximation to the GPE breaks down within $|z - z_{jk}| \sim \xi/b$, in these small regions we must solve the time-dependent GPE using a different perturbation theory, based on Taylor-expanding V about z_{jk}. Matching the hydrodynamic and core solutions smoothly together (see Refs. [9, 11]) finally yields, to leading order in ε,

$$\begin{aligned} i\rho\dot{D} \;=\; & \frac{1}{2}\left[\partial_{\bar{z}}\left(\rho\partial_z D + \rho\partial_{\bar{z}}\bar{D}\right) - \partial_z\left(\rho\partial_{\bar{z}}D\right)\right] - \partial_{\bar{z}}\left(\rho F_c\right) \\ & + \left(\ln\frac{b}{2\pi\xi} + 1.17\right)\partial_{\bar{z}}\rho - 0.20\partial_z\frac{\left(\partial_{\bar{z}}\rho\right)^2}{\rho} \end{aligned} \tag{12}$$

which is expressed in the co-rotating frame.

Eqns. (11) and (12) are our main results. The numerical co-efficients in (12) include some numerically evaluated contributions from the nonlinear core regions (compare with [11]), and also functions of τ, generally related to θ-functions, evaluated for the triangular case $\tau = (1 + i\sqrt{3})/2$. For general τ (*i.e.* for lattices other than the regular triangular), (12) would have several additional terms, such as $B(\tau)\partial_{\bar{z}\bar{z}}D$, where $B(\tau)$ is another

rapidly converging series. Modular covariance (a general type of lattice symmetry) of the extra co-efficients like $B(\tau)$ constrains them to vanish when $\tau = \frac{1+i\sqrt{3}}{2}$.

MSA implies that the lattice scale b is to the vortex matter equations much as the healing length ξ is to the hydrodynamic equations that underly them. Thus Eqns. (11) and (12) should be accurate except at distances of order b or less from of the edge of a vortex array. But is this claim really compatible with the results of Campbell and Ziff [3] for finite vortex arrays in infinite homogeneous superfluid? Setting ρ constant, to leading order in ε^2 the pair (11) (12) reduces to (2) and (3), and setting $\dot{D} \to \dot{D}_0 = 0$, we find a wealth of solutions to these fourth order equations:

$$D_0 = \sum_{m=1}^{\infty} \left(a_m \bar{z}^{m-1} + (m+1) b_m \bar{z}^m z - b_m^* z^{m+1} \right) \quad (13)$$

for arbitrary complex constants a_m, b_m. *A priori* it is unclear what boundary condition D should respect at the array edge; however if we assume that the edge should be a uniform circle of vortices, fitting leads to a unique combination of multipolar distortions with $m = 6, 12, 18 ....$ Figure 2 shows that stopping at only $m = 12$ gives quite good agreement with the first excited state for 217 vortices found in Ref. [3]. (The ground state differs only in the outermost ring.)

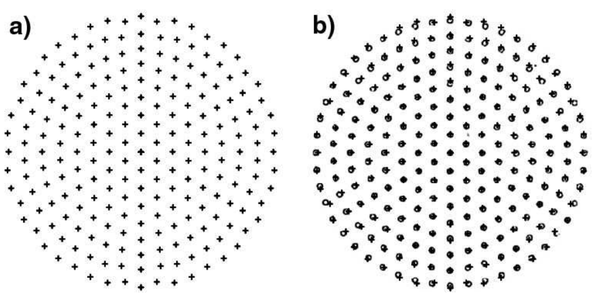

FIG. 2: Distortions of a finite vortex array in infinite homogeneous superfluid. Fig. 2a) shows the combination of $m = 0, 6$, and 12 solutions that makes the outer ring most circular and evenly spaced, while 2b) is a) overlaid on array 217_2 of Fig. 5 of Ref. [3].

If we set $\rho = \rho_0(1 - \varepsilon^2|z|^2)$ for a BEC in a round harmonic trap, though, the only distortion forced on us by the inhomogeneity is the very mild

$$D_0 = \frac{\sqrt{3}}{4\pi} \frac{\varepsilon^2 z}{1 - \varepsilon^2|z|^2} \left(\ln \frac{b}{2\pi\xi} + 1.17 \right) + \mathcal{O}\left(\varepsilon^4\right) \quad (14)$$

This scarcely visible radial shift of each vortex is shown in Fig. 1, for $\varepsilon = 1/6 = \xi(0)/b$. For vortices very close to the TF surface where formally $\xi \to \infty$, (14) spuriously predicts large inward displacements. (Four such vortices have been excised from Fig. 1a.) Apart from this failure in the lattice-edge boundary layer of thickness b, the accord with experiment is excellent.

Using (9) and (12), the v_c associated with this D_0 is

$$v_{c0} = iz \left(\Omega - \frac{\varepsilon^2 [\ln \frac{b}{2\pi\xi} + 1.17]}{1 - \varepsilon^2|z|^2} \right). \quad (15)$$

This purely azimuthal flow is slightly less than rigid body rotation at Ω, and the magnitude of the backflow increases with radius. This agrees qualitatively with the numerical results shown in Fig. 5 of Ref. [12].

Any static distortions forced by boundary conditions, like those of Fig. (2), would appear as zero-frequency modes among the collective excitations; so to these we now turn. We write $D = D_0 + \partial_{\bar{z}}(P + iQ)$, without loss of generality, for real P, Q. Then $\mathrm{Re}\{\partial_z[(12)/\rho]\}$ and (11) yield

$$\partial_{\bar{z}z} \dot{Q} = -2\Omega \partial_{\bar{z}z} P \times \left[1 + \mathcal{O}(\varepsilon^2)\right]. \quad (16)$$

Since only D is physical, any terms in P annihilated by $\partial_{\bar{z}z}$ are of form $f(z) + f^*(\bar{z})$ and so can be absorbed in Q as $i[f^*(\bar{z}) - f(z)]$. And since one can easily show that Laplacian-free terms in Q must be time-independent to leading order in ε^2, we can set $P = -\frac{1}{2\Omega}\dot{Q}$.

Introducing polar co-ordinates $re^{i\phi} = \varepsilon z$, so that $\rho = \rho_0(1-r^2)$, we can write $Q = q_{mn}(r) \cos(m\phi - \omega_{mn}t - \alpha)$ (for arbitrary constant α and angular and radial quantum numbers m and n). Considering first the case $m = 0$, where $\partial_\phi F_c = 0$, the imaginary part of $e^{-i\phi} \times (12)$ implies

$$\frac{4\omega_{0n}^2}{\varepsilon^2\Omega} \partial_r q_{0n} = -\frac{\left(\partial_r + 2r^{-1}\right)}{(1 - r^2)} \left[\left(1 - r^2\right)\left(\partial_r - r^{-1}\right)\partial_r q_{0n}\right] \quad (17)$$

plus order ε. Frobenius analysis shows that only one solution to this second order equation for $\partial_r q_{0n}$ is finite at $r \to 0$, and imposing finiteness at $r \to 1$ as well fixes the discrete spectrum. (Modified behavior within the boundary layer $r \gtrsim 1 - \varepsilon$ will be able to reconcile our coarse-scale D and v_c with microscopic boundary conditions, as long as our functions remain regular; compare the hydrodynamic derivation of collective modes in the vortex-free BEC [13].) Numerical search yields

$$\omega_{0n} = \varepsilon\Omega \{0, 1.43, 2.32, 3.18, 4.03, ...\}. \quad (18)$$

The zero mode is global rotation of the lattice, $q_{00} = r^2$.

These results are unsurprising for Tkachenko waves in a finite cylindrical system; but for $m \neq 0$, the dynamics is very different. To eliminate F_c we must take $\mathrm{Im}[\partial_z(12)]$, and the non-constant ρ leaves a first order time derivative on the left side, implying ω_{mn} of order ε^2 instead of ε. Our leading order equation is thus first order in t but fourth in r:

$$\begin{aligned} \frac{16m\omega_{mn}}{\varepsilon^2} q_{mn} &= -\left[\left(\partial_{rr} + r^{-1}\partial_r - m^2 r^{-2}\right) - 4r\partial_r\right] \\ &\quad \times \left(\partial_{rr} + r^{-1}\partial_r - m^2 r^{-2}\right) q_{mn}. \end{aligned} \quad (19)$$

This equation is quite singular (Frobenius analysis shows that of the four solutions only one is finite at both $r = 0, 1$); but its differential operator is Hermitian and self-dual, and its regular solution has a rapidly converging Frobenius series in r. From (12) we see that the radial component of v_c blows up at $r = 1$ unless $\left(\partial_{rr} - r^{-1}\partial_r - m^2 r^{-2}\right) q_{mn}(1) = 0$. This boundary condition on the regular solutions $q_{mn}(r)$ fixes the discrete spectrum, which may be found by summing Frobenius series numerically:

$\frac{\omega_{mn}}{\varepsilon^2\Omega}$	$m = 1$	2	3	4	5	6
$n = 0$	0	0.365	0.900	1.60	2.46	3.49
1	2.93	5.02	7.53	10.5	13.8	17.5
2	32.0	31.3	35.6	41.6	48.8	56.8
3	130.	105.	106.	113.	124.	137.

and so on. The translational zero mode is $q_{10} = r$. These eigenvalues of a fourth order equation increase rapidly with radial quantum number n, and as the co-efficients reach order ε^{-1} (19) becomes invalid, and WKB-like Tkachenko waves will emerge instead.

The only zero-frequency solution satisfying the boundary conditions is the rigid translation; and so (14) is the full static distortion. This reflects the fact that ρ is so small at the edges of the lattice that no boundary energies are large enough to influence the bulk lattice.

Only positive m have been reported, because $\omega_{-m,n} = -\omega_{mn}$ and replacing $m \to -m$ leaves our ansatz for Q unchanged. What this means is that the nonuniform ρ has drastically split the degeneracy of the two modes that would, for constant ρ, be proportional to $e^{\pm im\phi}$, with frequency of order ε. The linear combination proportional to $\cos(m\phi - \omega_{mn}t)$ propagates much more slowly, in the co-rotating frame; evidently the orthogonal combination, which we have not examined, propagates much more quickly. (The distortion patterns rotate about the origin; the vortices follow elliptical orbits about their equilibrium positions: see Fig. 3.) Thus the lowest frequency lattice modes have a first order dynamics, and only half as many distinct modes as for constant ρ.

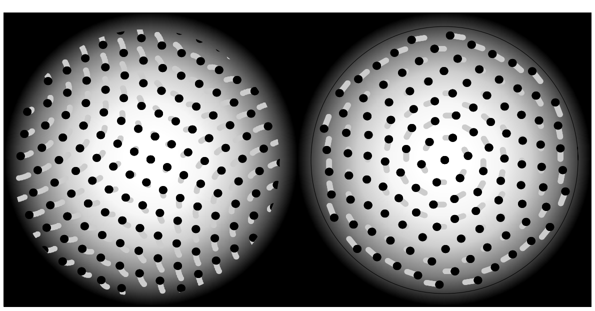

FIG. 3: Vortex lattice excitations in a round harmonic trap. The grey 'trails' indicate vortex motion. Left: $m, n = 2, 0$; compare Figure 3 b) of Ref. [2]. Right: $m, n = 0, 1$, in which motion is almost purely angular, and much faster.

Finally, note that for the quadrupole mode $\omega_{20} \neq 0$ (the zero eigenvalue solution $\tilde{q}_{20} = r^2$ does not satisfy the boundary condition). Since it is this mode which would distort the equilateral triangular lattice into the moderately different regular lattices that are also dynamically stable on short wavelengths [5], we conclude that although stable in bulk those lattices are frustrated in the finite system, and cannot even be stationary without slow but cumulatively large distortions. Reviewing our calculations in this context, it is clear that the only reason the equilateral triangular lattice does not suffer a similar fate is the vanishing, due to lattice symmetry, of several awkward terms from (12). Once the threat of cumulative distortion is lifted, it is not surprising that merely local distortion is of order ε^2. So the regularity of the observed BEC vortex lattices is ultimately due to their triangular structure. An engineer would attribute this to triangular rigidity, and a mathematician to the fact that the triangular lattice is the Z_3 fixed point of the modular group. Physicists are entitled to arbitrary linear combinations of the two explanations.

This work has been supported by the US National Science Foundation through a grant for the Center for Ultracold Atoms and through a grant for the Institute for Theoretical Atomic and Molecular Physics at Harvard University and the Smithsonian Astrophysical Observatory. MC acknowledges partial support from the DOE EPSCoR grant DE-FG01-00ER45832, Research Corporation Cottrell Science Award #CC5285, and a YSU Research Professorship Award.

[1] J.R. Abo-Shaeer et al., Science **292**, 476 (2001).
[2] P. Engels, I. Coddington, P. C. Haljan, and E. A. Cornell, cond-mat/0204449.
[3] L.J. Campbell and R.M. Ziff, Phys. Rev. **B20**, 1886 (1979).
[4] R.P. Feynman, Ch. 2 in C.J. Gorter, ed., *Progress in Low Temperature Physics* (North-Holland, Amsterdam, 1955).
[5] V.K. Tkachenko, Sov. Phys. JETP **23**, 1049 (1966); **29**, 945 (1969).
[6] G. Baym and E. Chandler, J. Low Temp. Phys. **50**, 57 (1983).
[7] M.R. Williams and A.L. Fetter, Phys. Rev. **B16**, 4846 (1977).
[8] V. L. Ginzburg and L. P. Pitaevskii, Sov. Phys. JETP **7**, 858 (1958); E. P. Gross, J. Math. Phys. **4**, 195 (1963).
[9] J.R. Anglin, Phys. Rev. **A65**, 063611 (2002).
[10] See, e.g., C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978), Chapter 11.
[11] B. Y. Rubinstein and L. M. Pismen, Physica D, **78**, 1 (1994).
[12] D. L. Feder and C. W. Clark, Phys. Rev. Lett. **87** 190401 (2001).

[13] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).